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On the structure of $U_q(sl(m,1))$: crystal bases

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Abstract. The structure of the quantized enveloping algebra of the Lie superalgebra $sl(m,1)$ is studied, crystal bases for its negative part are constructed.

1. Introduction

In the application of quantized enveloping algebras of Lie algebras (or Lie superalgebras) to two-dimensional solvable lattice models, the deformation parameter q essentially plays the role of temperature. At $q = 0$, which corresponds to absolute zero temperature, these algebras possess certain canonical bases called crystal bases introduced by Kashiwara [Kas1]. Crystal bases can be constructed for the quantized enveloping algebra of an arbitrary symmetrizable Kac–Moody algebra [Kas2]. Since crystal bases bear many remarkable properties (see [Kas1–Kas4]), there have been recent attempts to extend crystal base theory to the superalgebra case (see [BKK, MZ, Z1]).

In [Kas2], crystal bases for the negative part $\mathcal{U}^-$ of a quantized enveloping algebra $\mathcal{U}$ were constructed. These crystal bases have the property that when they are applied to a highest-weight vector of a simple $\mathcal{U}$-module V (of certain type), the non-vanishing elements form a crystal base for V. The construction uses the action of a reduced form of the quantized enveloping algebra. Similar reduced forms can also be defined for the superalgebra case (for the case $sl(m,n)$, see [Z1]). In this paper, we shall construct certain bases for the negative part $\mathcal{U}^-$ of the quantized enveloping algebra $\mathcal{U}$ of the Lie superalgebra $G = sl(m,1)$. Since these bases are crystal bases in the sense of [Kas2, 3.5] for the subalgebra $U_q(G_0) \cong U_q(gl(m))$, where G_0 is the even part of G, we shall also call them crystal bases. We first use the results on the reduced version $\mathcal{B}$ of $\mathcal{U}$ obtained in [Z1, Z2] to analyse the structure of $\mathcal{U}^-$ as a $\mathcal{B}$-module, then apply the result of [Kas2, 3.5] to $U_q(G_0)$ to construct these bases. The construction leads to two types of bases, which will be called the upper case crystal base and lower case crystal base, respectively.

In [BKK], a crystal base theory was developed for the Lie superalgebra $gl(m,n)$ for the category of modules obtained from the tensor products of the natural vector module of $gl(m,n)$. The crystal bases constructed in [BKK] are invariant under all Kashiwara operators and behave well with respect to tensor products. Due to the fact that the category of finite-dimensional $gl(m,n)$-modules is not completely reducible, a canonical base for $\mathcal{U}^-$ which is invariant under all Kashiwara operators does not seem to be possible. Nevertheless, the result of the present paper shows that some naturally constructed bases of $\mathcal{U}^-$ are actually crystal bases for the subalgebra $U_q(G_0)$ of $\mathcal{U}$.

In section 2, we will recall the definition of $\mathcal{U}$ and its reduced form $\mathcal{B}$, and study some basic properties of $\mathcal{U}$. In sections 3 and 4 we will construct crystal bases for $\mathcal{U}^-$.

2. The algebra $\mathcal{U}$ and its reduced form $\mathcal{B}$

As a contragredient algebra, the Lie superalgebra $G = sl(m, 1)$ $(m \geqslant 2)$ has the $m \times m$ defining matrix

$$(a_{ij})_{m\times m} = \begin{pmatrix} A_{m-1} & & \\ & & -1 \\ & -1 & 0 \end{pmatrix}$$

where A_{m-1} is the $(m-1)\times(m-1)$ Cartan matrix of type $\boldsymbol{A}$.

To define the quantized enveloping algebra $\mathcal{U}$ of G, let $\mathbb{C}$ be the field of complex numbers, let q be an indeterminate over $\mathbb{C}$, let $\mathcal{A}$ be the localization of the ring $\mathbb{C}[q]$ at $q = 0$ and let $\mathcal{F} = \mathbb{C}(q)$. The algebra $\mathcal{U}$ is an associative $\mathbb{Z}_2$-graded algebra over $\mathcal{F}$ (with 1) generated by e_i, f_i, $k_i^{\pm 1}$ $(i = 1, \ldots, m)$, with the grading given by $\deg(e_i) = \deg(f_i) = 0$ $(i = 1, \ldots, m-1)$, $\deg(k_i^{\pm 1}) = 0$ $(i = 1, \ldots, m)$, $\deg(e_m) = \deg(f_m) = 1$, and the defining relations

$$k_i k_i^{-1} = k_i^{-1} k_i = 1 \qquad k_i k_j = k_j k_i \qquad 1 \leqslant i, j \leqslant m \tag{2.1}$$

$$k_i e_j k_i^{-1} = q^{a_{ij}} e_j \qquad k_i f_j k_i^{-1} = q^{-a_{ij}} f_j \qquad 1 \leqslant i, j \leqslant m \tag{2.2}$$

$$e_i f_j - (-1)^{\deg(e_i)\deg(f_j)} f_j e_i = \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}} \qquad 1 \leqslant i, j \leqslant m \tag{2.3}$$

$e_i e_j = e_j e_i$ if $|i - j| > 1$, and for $|i - j| = 1$, $i \neq m$,

$$e_i^2 e_j - \left(q + q^{-1}\right) e_i e_j e_i + e_j e_i^2 = 0 \tag{2.4}$$

$f_i f_j = f_j f_i$ if $|i - j| > 1$, and for $|i - j| = 1$, $i \neq m$,

$$f_i^2 f_j - \left(q + q^{-1}\right) f_i f_j f_i + f_j f_i^2 = 0 \tag{2.5}$$

$$e_m^2 = 0 \qquad f_m^2 = 0. \tag{2.6}$$

As in [Kas2, 1.4], we can define two comultiplications $\Delta_\pm$ on $\mathcal{U}$. The comultiplication Δ_+ is defined by

$$\begin{aligned} &\Delta_+\left(k_i^{\pm 1}\right) = k_i^{\pm 1} \otimes k_i^{\pm 1} \\ &\Delta_+(e_i) = e_i \otimes 1 + k_i \otimes e_i \\ &\Delta_+(f_i) = f_i \otimes k_i^{-1} + 1 \otimes f_i \qquad 1 \leqslant i \leqslant m \end{aligned} \tag{2.7}$$

and the comultiplication Δ_- is defined by

$$\begin{aligned} &\Delta_-\left(k_i^{\pm 1}\right) = k_i^{\pm 1} \otimes k_i^{\pm 1} \\ &\Delta_-(e_i) = e_i \otimes k_i^{-1} + 1 \otimes e_i \\ &\Delta_-(f_i) = f_i \otimes 1 + k_i \otimes f_i \qquad 1 \leqslant i \leqslant m. \end{aligned} \tag{2.8}$$

Corresponding to $\Delta_\pm$, the antipodes of $\mathcal{U}$ are defined by

$$S_+(k_i) = k_i^{-1} \qquad S_+(e_i) = -k_i^{-1} e_i \qquad S_+(f_i) = -f_i k_i \qquad 1 \leqslant i \leqslant m \tag{2.9}$$

and (compare with [Jan, 9.13])

$$S_-(k_i) = k_i^{-1} \qquad S_-(e_i) = -e_i k_i \qquad S_-(f_i) = -k_i^{-1} f_i \qquad 1 \leqslant i \leqslant m \tag{2.10}$$

respectively.

We denote by $U_q(G_0)$ the subalgebra of $\mathcal{U}$ generated by $k_i^{\pm 1}$ $(1 \leqslant i \leqslant m)$, e_i and f_i $(1 \leqslant i \leqslant m-1)$. Since $G_0 \cong gl(m)$ as a Lie algebra, $U_q(G_0)$ is the usual quantized enveloping algebra of $gl(m)$. We denote by $\mathcal{U}^-$ (respectively $\mathcal{U}^+$) the subalgebra of $\mathcal{U}$ generated by f_i (respectively e_i), $1 \leqslant i \leqslant m$, and denote by $\mathcal{U}^0$ the subalgebra of $\mathcal{U}$ generated by $k_i^{\pm 1}$, $1 \leqslant i \leqslant m$. It is known that $\mathcal{U} = \mathcal{U}^- \mathcal{U}^0 \mathcal{U}^+$.

The reduced form $\mathcal{B}$ of $\mathcal{U}$ is the $\mathbb{Z}_2$-graded associative $\mathcal{F}$-algebra generated by e_i', f_i $(1 \leqslant i \leqslant m)$ with grading given by $\deg e_i' = \deg f_i = 0$ $(i \neq m)$, $\deg e_m' = \deg f_m = 1$, and generating relations

$$e_i' f_j = (-1)^{ab} q^{-a_{ij}} f_j e_i' + \delta_{ij} \qquad a = \deg e_i' \qquad b = \deg f_j \qquad 1 \leqslant i \quad j \leqslant m \tag{2.11}$$

$e_i' e_j' = e_j' e_i'$ if $|i-j| > 1$, and if $|i-j| = 1$, $i \neq m$

$$(e_i')^2 e_j' - \left(q + q^{-1}\right) e_i' e_j' e_i' + e_j' (e_i')^2 = 0 \tag{2.12}$$

$f_i f_j = f_j f_i$ if $|i-j| > 1$, and if $|i-j| = 1$, $i \neq m$

$$f_i^2 f_j - \left(q + q^{-1}\right) f_i f_j f_i + f_j f_i^2 = 0 \tag{2.13}$$

$$(e_m')^2 = 0 \qquad f_m^2 = 0. \tag{2.14}$$

The following lemma is proved in [Z1, Z2].

Lemma 2.1. *For any homogeneous element $y \in \mathcal{U}^-$ of $\mathbb{Z}_2$-grading b and any $1 \leqslant i \leqslant m$, there are unique y_i and y_i' in $\mathcal{U}^-$ such that*

$$e_i y - (-1)^{ab} y e_i = \frac{k_i y_i - y_i' k_i^{-1}}{q - q^{-1}} \tag{2.15}$$

where $a = \deg e_i$.

Thus we can define endomorphisms $e_i' : \mathcal{U}^- \to \mathcal{U}^-$, $1 \leqslant i \leqslant m$, by

$$e_i'(y) = k_i y_i' k_i^{-1} \tag{2.16}$$

where y_i' is given by lemma 2.1. If we also view f_i, $1 \leqslant i \leqslant m$, as endomorphisms of $\mathcal{U}^-$, then e_i' and f_i satisfy the defining relations of $\mathcal{B}$.

Lemma 2.2. *For any e_i' (considered as an endomorphism of $\mathcal{U}^-$), we have*

$$e_i'(u_1 u_2) = e_i'(u_1) u_2 + (-1)^{ab} k_i u_1 k_i^{-1} e_i'(u_2) \tag{2.17}$$

where $u_1, u_2 \in \mathcal{U}^-$ are homogeneous elements, $a = \deg(e_i')$, and $b = \deg(u_1)$.

Proof. Use lemma 2.1 and the definition of e_i'. □

By using arguments similar to those in the proofs of lemmas 3.4.2 and 3.4.3 of [Kas2], we can prove the following lemma.

Lemma 2.3. *The algebra $\mathcal{U}^-$ is a left $\mathcal{B}$-module and*

$$\mathcal{U}^- \cong \mathcal{B} \Big/ \sum_{i=1}^{m} \mathcal{B} e_i'.$$

We need to construct some root vectors of $\mathcal{U}$. As in [Kac1, 2.5.4], we use linear functions ϵ_i $(1 \leqslant i \leqslant m)$ and δ_1 to express the roots of $G = sl(m, 1)$ and choose

$$\left\{\alpha_i = \epsilon_i - \epsilon_{i+1}, 1 \leqslant i \leqslant m-1; \beta_m = \epsilon_m - \delta_1\right\}$$

as a simple root system. Let $\beta_i = \epsilon_i - \delta_1$, $1 \leqslant i \leqslant m$. Then the set of odd roots of G is $\{\pm\beta_i : 1 \leqslant i \leqslant m\}$. Let

$$Q^- = \sum_{i=1}^{m-1} a_i\alpha_i + a_m\beta_m \qquad a_i \in \{0, 1, 2, 3, \ldots\} \qquad 1 \leqslant i \leqslant m.$$

Then we have the usual weight subspaces decomposition

$$\mathcal{U}^- = \sum_{\lambda \in Q^-} \mathcal{U}^-_\lambda.$$

Recall that for a quantized enveloping algebra U of a Lie superalgebra with comultiplication Δ and antipode S, the adjoint action is defined by

$$ad_q x(y) = \sum (-1)^{\deg(a)\deg(y)} a y S(b) \tag{2.18}$$

where x and y are homogeneous elements of U and $\Delta x = \Sigma a \otimes b$. It is easy to verify that the $\mathbb{C}$-linear map $\theta : \mathcal{U} \to \mathcal{U}$ defined by

$$\theta e_i = f_i \qquad \theta f_i = e_i \qquad \theta k_i = k_i^{-1} \qquad \theta q = q^{-1} \tag{2.19}$$

and $\theta(uv) = \theta(v)\theta(u)$, $u, v \in \mathcal{U}$, is a $\mathbb{C}$-algebra anti-automorphism of $\mathcal{U}$.

Now we use (Δ_+, S_+) and (Δ_-, S_-) to construct two sets of odd root vectors of $\mathcal{U}$, respectively.

Define the following root vectors of $\mathcal{U}^-$:

$$\begin{aligned}
f_{\beta_m} &= f_m \\
f_{\beta_{m-1}} &= q^{-1} f_m f_{m-1} - f_{m-1} f_m \\
f_{\beta_{m-2}} &= q^{-1} f_{\beta_{m-1}} f_{m-2} - f_{m-2} f_{\beta_{m-1}} \\
&\ldots \\
f_{\beta_1} &= q^{-1} f_{\beta_2} f_1 - f_1 f_{\beta_2}.
\end{aligned} \tag{2.20}$$

Note that if we set $\theta(f_{\beta_i}) = e_{\beta_i}$, $1 \leqslant i \leqslant m$, then we have

$$e_{\beta_{m-i}} = q\, ad_q e_{m-i}\left(e_{\beta_{m-i+1}}\right) \qquad 1 \leqslant i \leqslant m-1 \tag{2.21}$$

with the adjoint action corresponding to (Δ_+, S_+).

By using (Δ_-, S_-), we define a different set of negative odd root vectors $f^-_{\beta_i}$, $1 \leqslant i \leqslant m$, as the following:

$$f^-_{\beta_m} = f_m \qquad f^-_{\beta_{m-1}} = ad_q f_{m-1}(f_m) \qquad \ldots \qquad f^-_{\beta_1} = ad_q f_1\left(f^-_{\beta_2}\right). \tag{2.22}$$

Note that by definition,

$$f^-_{\beta_i} = f_i f^-_{\beta_{i+1}} - q f^-_{\beta_{i+1}} f_i \qquad 1 \leqslant i \leqslant m-1. \tag{2.23}$$

3. Lower case crystal base

We have the following lemma.

Lemma 3.1. *For* $1 \leqslant i \leqslant m-1$, *we have*

$$e_i'\big(f^-_{\beta_{i_1}} f^-_{\beta_{i_2}} \cdots f^-_{\beta_{i_k}}\big) = 0 \qquad 1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant m \qquad 1 \leqslant k \leqslant m.$$

Proof. Since e_i commutes with 1, by the definition of e_i' we have $e_i'(1) = 0$. By (2.3) and (2.15) we have

$$e_i'(f_j) = \delta_{ij} \qquad 1 \leqslant i \quad j \leqslant m. \tag{3.1}$$

We claim that

$$e_i'\big(f^-_{\beta_j}\big) = 0 \qquad 1 \leqslant i \leqslant m-1 \quad 1 \leqslant j \leqslant m. \tag{3.2}$$

In fact, by (2.3), (2.17) and (3.1) we have $e_i'(f^-_{\beta_j}) = 0$ for $j > i$, hence use (2.23), we have

$$\begin{aligned} e_i'\big(f^-_{\beta_i}\big) &= e_i'(f_i) f^-_{\beta_{i+1}} - q k_i f^-_{\beta_{i+1}} k_i^{-1} e_i'(f_i) \\ &= f^-_{\beta_{i+1}} - q q^{-1} f^-_{\beta_{i+1}} \\ &= 0. \end{aligned}$$

By using induction on k and (2.17), it is clear that $e_i'(f^-_{\beta_{i-k}}) = 0$, $0 < k < i$, and therefore (3.2) follows. Now (2.17) and (3.2) imply the desired result. $\square$

Let

$$X^- = \big\{ f^-_{\beta_{i_1}} f^-_{\beta_{i_2}} \cdots f^-_{\beta_{i_k}} : 1 \leqslant k \leqslant m, 1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant m \big\} \bigcup \{1\}.$$

For $1 \leqslant i \leqslant m-1$, let $\tilde{f}_i$ and $\tilde{e}_i$ be the operators defined in [Kas2, 3.5] and let

$$B'(\infty) = \big\{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_t} \cdot x : x \in X^-, 1 \leqslant i_1, \ldots, i_t \leqslant m-1 \big\}.$$

Let $\tilde{f}_m = f_m$, $\tilde{e}_m = e_m'$. Let $L_-(\infty)$ be the $\mathcal{A}$-submodule of $\mathcal{U}^-$ spanned by $B'(\infty)$ and let $B_-(\infty)$ be the image of $B'(\infty)$ in $L_-(\infty)/qL_-(\infty)$.

Theorem 3.2. *The pair* $(L_-(\infty), B_-(\infty))$ *has the following properties:*

(a) $L_-(\infty)$ *is a free* $\mathcal{A}$*-module and it generates* $\mathcal{U}^-$ *as a vector space over* $\mathcal{F}$.
(b) $L_-(\infty) = \oplus_{\lambda \in Q^-} L_-(\infty)_\lambda$, *where* $L_-(\infty)_\lambda = L_-(\infty) \cap \mathcal{U}^-_\lambda$.
(c) $\tilde{e}_i L_-(\infty) \subseteq L_-(\infty)$ *for* $1 \leqslant i \leqslant m$ *and* $\tilde{f}_i L_-(\infty) \subseteq L_-(\infty)$ *for* $1 \leqslant i \leqslant m-1$.
(d) $B_-(\infty) = \cup_{\lambda \in Q^-} B_-(\infty)_\lambda$ *is a basis of the vector space* $L_-(\infty)/qL_-(\infty)$ *over* $\mathbb{C}$, *where* $B_-(\infty)_\lambda = B_-(\infty) \cap (L_-(\infty)_\lambda / qL_-(\infty)_\lambda)$.
(e) $\tilde{e}_i B_-(\infty) \subseteq B_-(\infty) \cup (0)$ *for* $1 \leqslant i \leqslant m$ *and* $\tilde{f}_i B_-(\infty) \subseteq B_-(\infty) \cup (0)$ *for* $1 \leqslant i \leqslant m-1$.
(f) For any $1 \leqslant i \leqslant m-1$ *and* $b, b' \in B_-(\infty)$. $b = \tilde{e}_i b' \Leftrightarrow b' = \tilde{f}_i b$.

Proof. Except for the statements that $\tilde{e}_m L_-(\infty) \subseteq L_-(\infty)$ and $\tilde{e}_m B_-(\infty) \subseteq B_-(\infty) \cup (0)$, all assertions follow from [Kas2, 3.5]. Note that $\tilde{e}_m \tilde{f}_i = \tilde{f}_i \tilde{e}_m$ for $i < m$, so it is clear that we only need to check (recall that $\tilde{e}_m \cdot 1 = 0$)

$$\tilde{e}_m\big(f^-_{\beta_{i_1}} f^-_{\beta_{i_2}} \cdots f^-_{\beta_{i_k}}\big) \qquad 1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant m \quad 1 \leqslant k \leqslant m.$$

We use induction on k, first consider $\tilde{e}_m(f^-_{\beta_i})$. For $i = m$, $\tilde{e}_m(f^-_{\beta_m}) = 1$. We claim that $\tilde{e}_m(f^-_{\beta_i}) = 0$ for $i < m$. In fact, if $i = m-1$, by (2.17) and (3.1) we have

$$\begin{aligned}\tilde{e}_m\big(f^-_{\beta_{m-1}}\big) &= e'_m\big(f^-_{\beta_{m-1}}\big)\\ &= e'_m\big(f_{m-1}f_m - qf_mf_{m-1}\big)\\ &= e'_m\big(f_{m-1}\big)f_m + qf_{m-1}e'_m(f_m) - q\big(e'_m(f_m)f_{m-1} - f_me'_m\big(f_{m-1}\big)\big)\\ &= 0.\end{aligned}$$

Assuming $\tilde{e}_m(f^-_{\beta_i}) = 0$, then again by (2.17) and (3.1) we have

$$\tilde{e}_m\big(f^-_{\beta_{i-1}}\big) = \tilde{e}_m\big(f_{i-1}f^-_{\beta_i} - qf^-_{\beta_i}f_{i-1}\big) = 0.$$

Hence by (2.17) and induction on k,

$$\tilde{e}_m\big(f^-_{\beta_{i_1}}f^-_{\beta_{i_2}}\cdots f^-_{\beta_{i_k}}\big) = \begin{cases}0 & i_k < m\\ (-1)^{k-1}q^{k-1}f^-_{\beta_{i_1}}f^-_{\beta_{i_2}}\cdots f^-_{\beta_{i_{k-1}}} & i_k = m.\end{cases}$$

Therefore,

$$\tilde{e}_m\big(f^-_{\beta_{i_1}}f^-_{\beta_{i_2}}\cdots f^-_{\beta_{i_k}}\big) \in L_-(\infty)$$

and

$$\tilde{e}_m\big(f^-_{\beta_{i_1}}f^-_{\beta_{i_2}}\cdots f^-_{\beta_{i_k}}\big) = 0 \pmod{qL_-(\infty)}$$

which proves the statements about $\tilde{e}_m$. □

Remark. Since $f_mf_{m-1} = q^{-1}f_{m-1}f_m - q^{-1}f_{\beta_{m-1}}$, $\tilde{f}_m(L_-(\infty))$ is not included in $L_-(\infty)$.

4. Upper case crystal base

Similar to the proof of lemma 3.1 we can prove the following lemma.

Lemma 4.1. *For* $1 \leqslant i \leqslant m-1$, *we have*

$$e'_i\big(f_{\beta_{i_1}}f_{\beta_{i_2}}\cdots f_{\beta_{i_k}}\big) = 0 \qquad 1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant m \quad 1 \leqslant k \leqslant m.$$

Let

$$X = \big\{f_{\beta_{i_1}}f_{\beta_{i_2}}\cdots f_{\beta_{i_k}} : 1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant m, 1 \leqslant k \leqslant m\big\}\bigcup\{1\}$$

let $\tilde{f}_i, \tilde{e}_i$ $(1 \leqslant i \leqslant m)$ be the operators defined in section 3 and let

$$B''(\infty) = \big\{\tilde{f}_{i_1}\cdots\tilde{f}_{i_t}\cdot x : x \in X, 1 \leqslant i_1, \ldots, i_t \leqslant m-1\big\}.$$

Let $L(\infty)$ be the $\mathcal{A}$-submodule of $\mathcal{U}^-$ spanned by $B''(\infty)$ and let $B(\infty)$ be the image of $B''(\infty)$ in $L(\infty)/qL(\infty)$.

Theorem 4.2. *The pair* $(L(\infty), B(\infty))$ *has the following properties:*

(a) $L(\infty)$ *is a free* $\mathcal{A}$*-module and it generates* $\mathcal{U}^-$ *as a vector space over* $\mathcal{F}$.
(b) $L(\infty) = \oplus_{\lambda\in Q^-}L(\infty)_\lambda$, *where* $L(\infty)_\lambda = L(\infty)\cap\mathcal{U}^-_\lambda$.
(c) $\tilde{f}_iL(\infty) \subseteq L(\infty)$ *for* $1 \leqslant i \leqslant m$ *and* $\tilde{e}_iL(\infty) \subseteq L(\infty)$ *for* $1 \leqslant i \leqslant m-1$.

(d) $B(\infty) = \cup_{\lambda\in Q^-} B(\infty)_\lambda$ *is a basis of the vector space* $L(\infty)/qL(\infty)$ *over* $\mathbb{C}$, *where* $B(\infty)_\lambda = B(\infty) \cap (L(\infty)_\lambda/qL(\infty)_\lambda)$.
(e) $\tilde{f}_i B(\infty) \subseteq B(\infty) \cup (0)$ *for* $1 \leqslant i \leqslant m$ *and* $\tilde{e}_i B(\infty) \subseteq B(\infty) \cup (0)$ *for* $1 \leqslant i \leqslant m-1$.
(f) *For any* $1 \leqslant i \leqslant m-1$ *and* $b, b' \in B_-(\infty)$, $b = \tilde{e}_i b' \Leftrightarrow b' = \tilde{f}_i b$.

It is clear that we only need to prove $\tilde{f}_m L(\infty) \subseteq L(\infty)$ and $\tilde{f}_m B(\infty) \subseteq B(\infty) \cup (0)$. In order to do that, we need some commutation formulae which we will state as lemmas.

Lemma 4.3. *For* $i < m$, $f_m f_{\beta_i} = -q f_{\beta_i} f_m$.

Proof. We let $i = m-k$, and use induction on k. For $k=1$, use $f_m^2 = 0$, we have

$$\begin{aligned} f_m f_{\beta_{m-1}} &= f_m\big(q^{-1} f_m f_{m-1} - f_{m-1} f_m\big) = -f_m f_{m-1} f_m \\ &= -q\big(q^{-1} f_m f_{m-1} - f_{m-1} f_m\big) f_m = -q f_{\beta_{m-1}} f_m. \end{aligned}$$

Assume the formula is true for $k \geqslant 1$, since $f_m f_{m-k-1} = f_{m-k-1} f_m$, we have

$$\begin{aligned} f_m f_{\beta_{m-(k+1)}} &= f_m (q^{-1} f_{\beta_{m-k}} f_{m-k-1} - f_{m-k-1} f_{\beta_{m-k}}) \\ &= -f_{\beta_{m-k}} f_{m-k-1} f_m + q f_{m-k-1} f_{\beta_{m-k}} f_m \\ &= -q f_{\beta_{m-(k+1)}} f_m. \end{aligned}$$

Thus the lemma follows. □

Lemma 4.3 implies that for $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant m$,

$$f_m f_{\beta_{i_1}} f_{\beta_{i_2}} \cdots f_{\beta_{i_k}} = (-1)^k q^k f_{\beta_{i_1}} f_{\beta_{i_2}} \cdots f_{\beta_{i_k}} f_m. \tag{4.1}$$

Lemma 4.4. *For* $0 \leqslant i \leqslant m-1$, *we have*

$$f_{m-i-1}^2 f_{\beta_{m-i}} - (q+q^{-1}) f_{m-i-1} f_{\beta_{m-i}} f_{m-i-1} + f_{\beta_{m-i}} f_{m-i-1}^2 = 0. \tag{4.2}$$

Proof. For $i = 0$, the formula is just one of the defining relation of $\mathcal{U}$. For $i > 0$, we have

$$\begin{aligned} f_{m-i-1}^2 f_{\beta_{m-i}} &= f_{m-i-1}^2\big(q^{-1} f_{\beta_{m-i+1}} f_{m-i} - f_{m-i} f_{\beta_{m-i+1}}\big) \\ &= q^{-1} f_{\beta_{m-i+1}} f_{m-i-1}^2 f_{m-i} - f_{m-i-1}^2 f_{m-i} f_{\beta_{m-i+1}} \\ &= q^{-1} f_{\beta_{m-i+1}}\big(\big(q+q^{-1}\big) f_{m-i-1} f_{m-i} f_{m-i-1} - f_{m-i} f_{m-i-1}^2\big) \\ &\quad - \big(\big(q+q^{-1}\big) f_{m-i-1} f_{m-i} f_{m-i-1} - f_{m-i} f_{m-i-1}^2\big) f_{\beta_{m-i+1}} \\ &= q^{-1}\big(q+q^{-1}\big) f_{m-i-1} f_{\beta_{m-i+1}} f_{m-i} f_{m-i-1} - q^{-1} f_{\beta_{m-i+1}} f_{m-i} f_{m-i-1}^2 \\ &\quad - \big(q+q^{-1}\big) f_{m-i-1} f_{m-i} f_{\beta_{m-i+1}} f_{m-i-1} + f_{m-i} f_{\beta_{m-i+1}} f_{m-i-1}^2 \\ &= \big(q+q^{-1}\big) f_{m-i-1} f_{\beta_{m-i}} f_{m-i-1} - f_{\beta_{m-i}} f_{m-i-1}^2. \end{aligned}$$

Thus (4.2) follows. □

Formula (4.2) implies

$$f_{m-i} f_{\beta_{m-i}} = q f_{\beta_{m-i}} f_{m-i} \qquad 0 \leqslant i \leqslant m-1. \tag{4.3}$$

Lemma 4.5. *For* $0 \leqslant i \leqslant m-1$, *we have*

$$f_{\beta_{m-i}}^2 = 0. \tag{4.4}$$

Proof. We use induction on i. The case $i = 0$ follows from (2.6). Assume $f_{\beta_{m-i}}^2 = 0$, then a simple computation shows

$$f_{\beta_{m-i}} f_{\beta_{m-i-1}} = -q f_{\beta_{m-i-1}} f_{\beta_{m-i}}. \tag{4.5}$$

Thus

$$\begin{aligned} f_{\beta_{m-i-1}}^2 &= \left(q^{-1} f_{\beta_{m-i}} f_{m-i-1} - f_{m-i-1} f_{\beta_{m-i}}\right) f_{\beta_{m-i-1}} \\ &= -q f_{\beta_{m-i-1}} f_{\beta_{m-i}} f_{m-i-1} + q^2 f_{\beta_{m-i-1}} f_{m-i-1} f_{\beta_{m-i}} \end{aligned}$$

(by (4.3), (4.5))

$$= -q^2 f_{\beta_{m-i-1}}^2.$$

Therefore, by our assumption on q, the desired formula follows. □

Lemma 4.6. *For* $0 \leqslant i < j \leqslant m - 1$, *we have*

$$f_{\beta_{m-i}} f_{\beta_{m-j}} = -q f_{\beta_{m-j}} f_{\beta_{m-i}}. \tag{4.6}$$

Proof. Note that the case $i = 0$ is just lemma 4.3, and by (4.5) the formula is true for $j = i+1$. Assume it is true for $j = i + k$ with $k \geqslant 1$. Then since $f_{\beta_{m-i}} f_{m-i-k-1} = f_{m-i-k-1} f_{\beta_{m-i}}$, we have

$$\begin{aligned} f_{\beta_{m-i}} f_{\beta_{m-i-k-1}} &= f_{\beta_{m-i}}\left(q^{-1} f_{\beta_{m-i-k}} f_{m-i-k-1} - f_{m-i-k-1} f_{\beta_{m-i-k}}\right) \\ &= -q\left(q^{-1} f_{\beta_{m-i-k}} f_{m-i-k-1} - f_{m-i-k-1} f_{\beta_{m-i-k}}\right) f_{\beta_{m-i}} \\ &= -q f_{\beta_{m-i-k-1}} f_{\beta_{m-i}}. \end{aligned}$$

Hence the desired formula follows. □

Lemma 4.7. *For* $0 \leqslant i < m - 1$, *we have*

$$f_{\beta_{m-i}} f_{m-i-1}^{(k)} = q f_{m-i-1}^{(k-1)} f_{\beta_{m-i-1}} + q^k f_{m-i-1}^{(k)} f_{\beta_{m-i}} \tag{4.7}$$

where $f_j^{(k)} = f_j^k/[k]!$.

Proof. We use induction on k. We distinguish between two cases: $i = 0$ and $i > 0$. The case where $i = 0$ is straightforward, we give the proof for the case where $i > 0$. For $k = 1$, the desired formula follows from the definition of $f_{\beta_{m-i-1}}$ (see (2.20)). Assume the formula holds for $k - 1$, then

$$\begin{aligned} f_{\beta_{m-i}} f_{m-i-1}^{(k)} &= \frac{1}{[k]} f_{\beta_{m-i}} f_{m-i-1}^{(k-1)} f_{m-i-1} \\ &= \frac{1}{[k]}\left(q f_{m-i-1}^{(k-2)} f_{\beta_{m-i-1}} + q^{k-1} f_{m-i-1}^{(k-1)} f_{\beta_{m-i}}\right) f_{m-i-1} \\ &= \frac{1}{[k]} f_{m-i-1}^{(k-2)} f_{m-i-1} f_{\beta_{m-i-1}} + \frac{1}{[k]} q^{k-1} f_{m-i-1}^{(k-1)}\left(q f_{\beta_{m-i-1}} + q f_{m-i-1} f_{\beta_{m-i}}\right) \end{aligned}$$

(by (4.3))

$$\begin{aligned} &= \frac{1}{[k]}\left(f_{m-i-1}^{(k-2)} f_{m-i-1} + q^k f_{m-i-1}^{(k-1)}\right) f_{\beta_{m-i-1}} + q^k f_{m-i-1}^{(k)} f_{\beta_{m-i}} \\ &= \frac{1}{[k]}\left([k-1] + q^k\right) f_{m-i-1}^{(k-1)} f_{\beta_{m-i-1}} + q^k f_{m-i-1}^{(k)} f_{\beta_{m-i}} \\ &= q f_{m-i-1}^{(k-1)} f_{\beta_{m-i-1}} + q^k f_{m-i-1}^{(k)} f_{\beta_{m-i}} \end{aligned}$$

and the formula follows. □

Lemma 4.8. *For* $0 \leqslant i, j \leqslant m-1$, *we have*

$$f_{\beta_{m-i}} f_{m-j} = \begin{cases} f_{m-j} f_{\beta_{m-i}} & j \neq i \quad i+1 \\ q^{-1} f_{m-i} f_{\beta_{m-i}} & j = i \\ q f_{\beta_{m-i-1}} + q f_{m-i-1} f_{\beta_{m-i}} & j = i+1. \end{cases} \tag{4.8}$$

Proof. The case $j = i$ is just (4.3), and the case $j = i+1$ follows from (2.20), so we only need to prove the case $j \neq i, i+1$.

Assume $j \neq i, i+1$. The formula clearly holds if $j > i$. To treat the case $j < i$, we let $j = i - k$ and use induction on k. For $k = 1$, we have

$$\begin{aligned} f_{\beta_{m-i}} f_{m-i+1} &= \left(q^{-1} f_{\beta_{m-i+1}} f_{m-i} - f_{m-i} f_{\beta_{m-i+1}}\right) f_{m-i+1} \\ &= q^{-1}\left(q^{-1} f_{\beta_{m-i+2}} f_{m-i+1} - f_{m-i+1} f_{\beta_{m-i+2}}\right) f_{m-i} f_{m-i+1} - f_{m-i} f_{\beta_{m-i+1}} f_{m-i+1} \\ &= q^{-2} f_{\beta_{m-i+2}} f_{m-i+1} f_{m-i} f_{m-i+1} - q^{-1} f_{m-i+1} f_{\beta_{m-i+2}} f_{m-i} f_{m-i+1} \\ &\quad - q^{-1} f_{m-i} f_{m-i+1} f_{\beta_{m-i+1}} \qquad \text{(by (4.3))}. \end{aligned}$$

The first term at the last step is equal to

$$\begin{aligned} &\frac{q^{-2}}{q+q^{-1}} f_{\beta_{m-i+2}} \left(f_{m-i+1}^2 f_{m-i} + f_{m-i} f_{m-i+1}^2\right) \\ &\quad = \frac{q^{-2}}{q+q^{-1}} \left(q f_{\beta_{m-i+1}} + q f_{m-i+1} f_{\beta_{m-i+2}}\right) f_{m-i+1} f_{m-i} \\ &\qquad + \frac{q^{-2}}{q+q^{-1}} \left(q f_{\beta_{m-i+1}} + q f_{m-i+1} f_{\beta_{m-i+2}}\right) f_{m-i+1} \\ &\quad = \frac{q^{-2}}{q+q^{-1}} f_{m-i+1} f_{\beta_{m-i+1}} f_{m-i} \\ &\qquad + \frac{1}{q+q^{-1}} f_{m-i+1} \left(f_{\beta_{m-i+1}} + f_{m-i+1} f_{\beta_{m-i+2}}\right) f_{m-i} \\ &\qquad + \frac{q^{-2}}{q+q^{-1}} f_{m-i} f_{m-i+1} f_{\beta_{m-i+1}} \\ &\qquad + \frac{1}{q+q^{-1}} f_{m-i} f_{m-i+1} \left(f_{\beta_{m-i+1}} + f_{m-i+1} f_{\beta_{m-i+2}}\right) \\ &\qquad = f_{m-i+1}\left(f_{\beta_{m-i}} + f_{m-i} f_{\beta_{m-i+1}}\right) + \frac{1}{q+q^{-1}} f_{m-i+1}^2 f_{m-i} f_{\beta_{m-i+2}} \\ &\qquad + q^{-1} f_{m-i} f_{m-i+1} f_{\beta_{m-i+1}} + \frac{1}{q+q^{-1}} f_{m-i} f_{m-i+1}^2 f_{\beta_{m-i+2}} \end{aligned}$$

and the second term is equal to

$$\begin{aligned} &-f_{m-i+1} f_{m-i} \left(f_{\beta_{m-i+1}} + f_{m-i+1} f_{\beta_{m-i+2}}\right) \\ &\quad = -f_{m-i+1} f_{m-i} f_{\beta_{m-i+1}} - f_{m-i+1} f_{m-i} f_{m-i+1} f_{\beta_{m-i+2}} \end{aligned}$$

hence

$$f_{\beta_{m-i}} f_{m-i+1} = f_{m-i+1} f_{\beta_{m-i}}.$$

Suppose the desired formula holds for $k-1 \geqslant 1$, then by what we have proved,

$$\begin{aligned} f_{\beta_{m-i}} f_{m-i+k} &= \left(q^{-1} f_{\beta_{m-i+1}} f_{m-i} - f_{m-i} f_{\beta_{m-i+1}}\right) f_{m-i+k} \\ &= f_{m-i+k}\left(q^{-1} f_{\beta_{m-i+1}} f_{m-i} - f_{m-i} f_{\beta_{m-i+1}}\right) \\ &= f_{m-i+k} f_{\beta_{m-i}} \end{aligned}$$

so the lemma follows. □

Proof of theorem 4.2 To prove $\tilde{f}_m L(\infty) \subseteq L(\infty)$ and $\tilde{f}_m B(\infty) \subseteq B(\infty) \cup (0)$, we consider a typical element of the form

$$\tilde{f}_{i_1} \cdots \tilde{f}_{i_r} f_{\beta_{j_1}} \cdots f_{\beta_{j_s}} \cdot 1$$

and use induction on r. For $r = 0$, formula (4.1) implies that

$$\tilde{f}_m f_{\beta_{j_1}} \cdots f_{\beta_{j_s}} \cdot 1 \in L(\infty)$$

and

$$\tilde{f}_m f_{\beta_{j_1}} \cdots f_{\beta_{j_s}} \cdot 1 \equiv 0 \qquad (\mathrm{mod}\ qL(\infty)).$$

Consider the case $r = 1$. If $\tilde{f}_{i_1} \neq \tilde{f}_{m-1}$, then since f_m commutes with f_{i_1}, by the above arguments the desired result follows. If $\tilde{f}_{i_1} = \tilde{f}_{m-1}$, then by (4.8), we have

$$\tilde{f}_m \tilde{f}_{m-1} f_{\beta_{j_1}} \cdots f_{\beta_{j_s}} \cdot 1 = \left(q f_{\beta_{m-1}} + q \tilde{f}_{m-1} \tilde{f}_m\right) f_{\beta_{j_1}} \cdots f_{\beta_{j_s}} \cdot 1$$

and by (4.6) we have the desired result. So let us assume that

$$\tilde{f}_m \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} f_{\beta_{j_1}} \cdots f_{\beta_{j_s}} \cdot 1 \in \boldsymbol{B}(\infty) \cup (0) \tag{4.9}$$

and

$$q^i f_{\beta_{m-i}} \tilde{f}_{k_1} \cdots \tilde{f}_{k_{r-i}} f_{\beta_{j_1}} \cdots f_{\beta_{j_s}} \cdot 1 \in \boldsymbol{B}(\infty) \cup (0) \tag{4.10}$$

where $k_1 \neq m - i$. The assumption $k_1 \neq m - i$ is based on the fact that $f_{\beta_{m-i}}$ commutes with f_{m-j} for $j > i + 1$ and the following formula (see formula (1) on p 253 in [Jan])

$$f_i f_{i+1}^{(k)} = q^k f_{i+1}^{(k)} f_i + f_{i+1}^{(k-1)}\left(f_i f_{i+1} - q f_{i+1} f_i\right) \tag{4.11}$$

for $i < m - 1$.

Now consider

$$\tilde{f}_m \tilde{f}_{i_1} \cdots \tilde{f}_{i_{r+1}} f_{\beta_{j_1}} \cdots f_{\beta_{j_s}} \cdot 1.$$

Then since $\tilde{f}_m$ commutes with $\tilde{f}_{m-k}$ for $k > 1$, we can assume that $i_1 = m - 1$. By formula (4.7), we can further assume that $i_2 \neq m - 1$. Since

$$\tilde{f}_m \tilde{f}_{i_1} \cdots \tilde{f}_{i_{r+1}} f_{\beta_{j_1}} \cdots f_{\beta_{j_s}} \cdot 1 = \left(q f_{\beta_{m-1}} + q \tilde{f}_{m-1} \tilde{f}_m\right) \tilde{f}_{i_2} \cdots \tilde{f}_{i_{r+1}} f_{\beta_{j_1}} \cdots f_{\beta_{j_s}} \cdot 1$$

by our induction assumption (4.10), it is clear that we only need to consider

$$q f_m \tilde{f}_{i_2} \cdots \tilde{f}_{i_{r+1}} f_{\beta_{j_1}} \cdots f_{\beta_{j_s}} \cdot 1$$

and assume that $i_2 = m - 2 \neq i_3$ by reasons similar to those in the discussion before. Proceeding similarly, we reduce our case to the term

$$q^{m-1} f_{\beta_1} \tilde{f}_{i_m} \cdots \tilde{f}_{i_{r+1}} f_{\beta_{j_1}} \cdots f_{\beta_{j_s}} \cdot 1.$$

Since formula (4.11) allows us to assume $i_m \neq 1$, by (4.8), $f_{\beta_1} f_{i_m} = f_{i_m} f_{\beta_1}$, thus by induction assumption (4.10) the induction process goes through and the proof of theorem 4.2 has been completed. □

5. Discussion

We used the reduced form of the quantized enveloping algebra $\mathcal{U}$ of $G = sl(m,1)$ to analyse the structure of $\mathcal{U}$. Our result shows that some naturally constructed bases for $\mathcal{U}^-$ are crystal bases for the subalgebra of $\mathcal{U}$ corresponding to the even part of G in the sense of [Kas1]. In [Z2], these bases were shown to have the property that when applied to a highest-weight vector of a simple module (of certain type), the non-vanishing ones form a crystal base for the module (for the subalgebra corresponds to the even part of G) in the case of $sl(2,1)$. Further attention should be given for the general cases.

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